Derivation of the Shannon Spectral Efficiency Limit

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10–Apr–2014

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Typeset using $\LaTeX\ 2_\varepsilon$

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1 Introduction

In digital communication systems, *spectral efficiency*, η, is defined as

$$\eta = \frac{C}{W}, \tag{1}$$

where C is the channel capacity (in [bits/second]) and W is the channel bandwidth (in [Hz]).

Spectral efficiency provides a measure of how efficiently a communication scheme utilizes bandwidth. For any fixed capacity C, spectral efficiency approaches zero as the bandwidth approaches infinity. This is the absolute worst efficiency possible (infinite bandwidth!).

Shannon's famous capacity theorem can be rearranged to provide a relationship between spectral efficiency and SNR, or E_b/N_0[1] (see Figure 1). It appears from the plot that the relationship approaches an asymptotic limit as $\eta \to 0$. In this paper we show this limit is $\ln 2$, or approximately -1.59 dB.

2 Derivation

Shannon's capacity theorem is stated in [2] as follows:

Shannon's Capacity Theorem. *Let P be the average transmitter power, and suppose the noise is white thermal noise of power N in the band W. By sufficiently complicated encoding systems it is possible to transmit binary digits at a rate*

$$C = W \log_2\left(1 + \frac{P}{N}\right), \tag{2}$$

with as small a frequency of errors as desired. It is not possible by any encoding method to send at a higher rate and have an arbitrarily low frequency of errors.

If the energy per bit transmitted is denoted E_b and we are transmitting C bits per second, then the signal power $P = CE_b$. Since the noise is white, the noise power in a bandwidth W is WN_0 where N_0 is the noise spectral density, and we may rewrite the capacity equation as

$$\frac{C}{W} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{W}\right). \tag{3}$$

[1] For a good discussion on SNR and E_b/N_0 see section 5.2.2 in [1].

If we let $\eta = C/W$ (the spectral efficiency), then we can reexpress in terms of E_b/N_0 as

$$\frac{E_b}{N_0} = \frac{2^\eta - 1}{\eta}, \tag{4}$$

that is, we can express E_b/N_0 as a function of η, which is plotted in Figure 1.

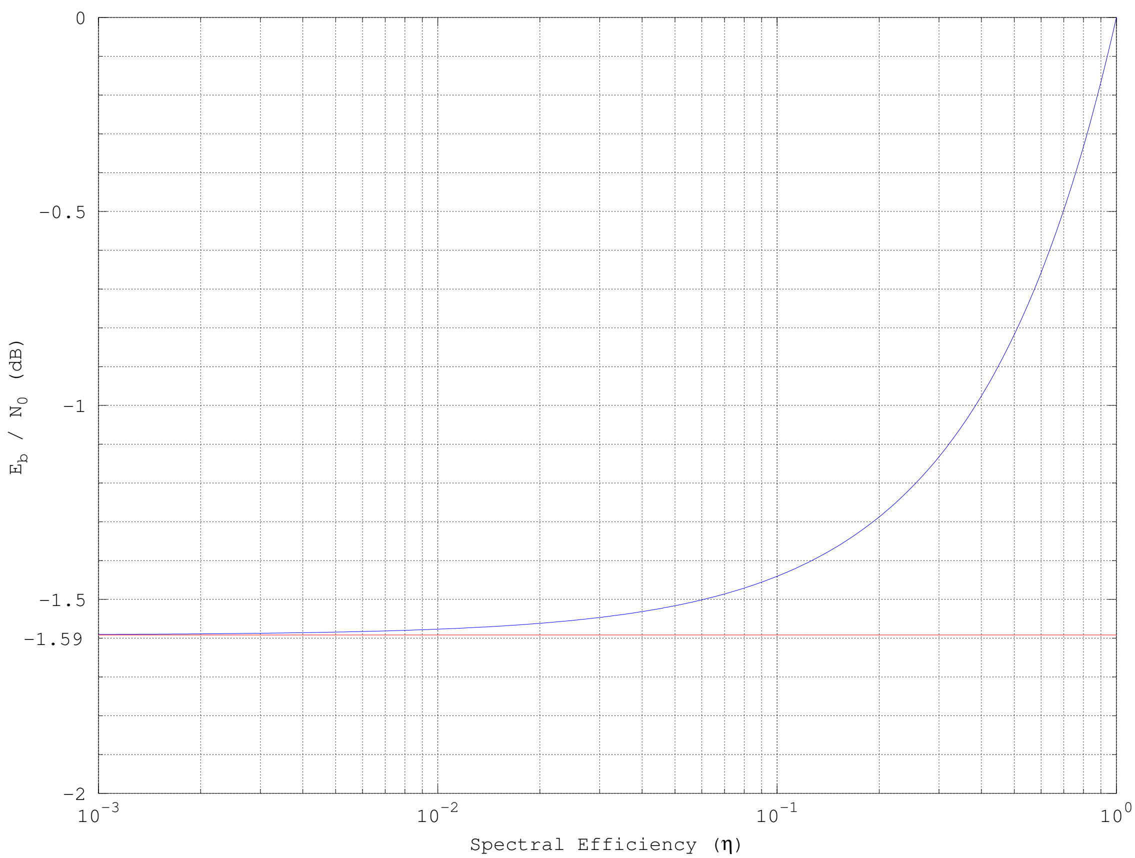

Figure 1: Capacity Curve

The red line is the asymptote of the curve as η approaches zero and is approximately -1.59 dB ($\ln 2$). We derive this result as follows.

The asymptotic value we seek, a, is the limit of E_b/N_o as η approachs zero.

$$a = \lim_{\eta \to 0} \frac{2^\eta - 1}{\eta}. \tag{5}$$

Since the argument of the limit is a rational function $f(\eta)/g(\eta)$ and $f(0) = g(0) = 0$, we may apply L'Hopital's rule, which states [3] that, under these conditions,

$$\lim_{\eta \to 0} \frac{f(\eta)}{g(\eta)} = \lim_{\eta \to 0} \frac{f'(\eta)}{g'(\eta)}. \tag{6}$$

In order to take the derivative of the numerator, express the value 2 in terms of the natural logarithm:

$$2 = e^{\ln 2}. \tag{7}$$

Then

$$\begin{align} 2^\eta &= \left(e^{\ln 2}\right)^\eta \tag{8}\\ &= e^{\eta \ln 2} \tag{9}\end{align}$$

and

$$\frac{\mathrm{d}2^\eta}{\mathrm{d}\eta} = \frac{\mathrm{d}e^u}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\eta}, \tag{10}$$

where $u = \eta \ln 2$. Therefore

$$\begin{align} a &= \lim_{\eta \to 0} \frac{\mathrm{d}2^\eta}{\mathrm{d}\eta} \tag{11}\\ &= \lim_{\eta \to 0} e^u \ln 2 \tag{12}\\ &= \lim_{\eta \to 0} \left(e^{\ln 2}\right)^\eta \ln 2 \tag{13}\\ &= \ln 2. \blacksquare \tag{14}\end{align}$$

3 Conclusion

In a digital communication system, we can trade off bandwidth (spectral efficiency) for E_b/N_0, as seen from Figure 1. For example, a spectral efficiency of one requires an E_b/N_0 of approximately 0 dB to reach capacity. But if we allow our spectral efficiency to be reduced to 0.1, we can reach the Shannon capacity limit with 1.45 dB less power.

However, this tradeoff has a limit in that, no matter how much bandwidth you allow a signal to have, you can never use less than -1.59 dB E_b/N_0 and still achieve capacity.

References

[1] Bernard Sklar, *Digital Communications*, 2nd ed. Prentice Hall P T R, 2001.

[2] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the Institute of Radio Engineers*, vol. 37, pp. 10–21, 1949.

[3] S. K. Stein, *Calculus and Analytic Geometry*, 2nd ed. McGraw-Hill, 1973.