Author	Date	Time	Rev	No.	Reference
Randy Yates	16-Feb-2006	23:20	PA1	n/a	shannonlimit.tex

Shannon's capacity theorem is stated in [1] as follows:

Shannon's Capacity Theorem. *Let P be the average transmitter power, and suppose the noise is white thermal noise of power N in the band W. By sufficiently complicated encoding systems it is possible to transmit binary digits at a rate*

$$C = W \log_2 \left(1 + \frac{P}{N}\right), \tag{1}$$

with as small a frequency of errors as desired. It is not possible by any encoding method to send at a higher rate and have an arbitrarily low frequency of errors.

If the energy per bit transmitted is denoted E_b and we are transmitting C bits per second, then the signal power $P = CE_b$. Since the noise is white, the noise power in a bandwidth W is WN_0 where N_0 is the noise spectral density, and we may rewrite the capacity equation as

$$\frac{C}{W} = \log_2 \left(1 + \frac{E_b}{N_0}\frac{C}{W}\right). \tag{2}$$

If we let $\eta = C/W$ (the spectral efficiency), then we can reexpress in terms of E_b/N_0 as

$$\frac{E_b}{N_0} = \frac{2^\eta - 1}{\eta}, \tag{3}$$

which is plotted in Figure 1.

The dotted red line is the asymptote of E_b/N_0 as the bandwidth W approaches infinity and is approximately -1.6 dB ($\ln 2$). We derive this result as follows.

The asymptotic value we seek, a, is the limit of E_b/N_o as η approachs zero.

$$a = \lim_{\eta \to 0} \frac{2^\eta - 1}{\eta}. \tag{4}$$

Since the argument of the limit is a rational function $f(\eta)/g(\eta)$ and $f(0) = g(0) = 0$, we may apply L'Hopital's rule.

In order to take the derivative of the numerator, express the value 2 in terms of the natural logarithm:

$$2 = e^{\ln 2}. \tag{5}$$

Then

$$2^\eta = \left(e^{\ln 2}\right)^\eta \tag{6}$$

$$= e^{\eta \ln 2} \tag{7}$$

and

$$\frac{d2^\eta}{d\eta} = \frac{de^u}{du}\frac{du}{d\eta}, \tag{8}$$

where $u = \eta \ln 2$. Therefore

$$a = \lim_{\eta \to 0} \frac{d2^\eta}{d\eta} \tag{9}$$

$$= \lim_{\eta \to 0} e^u \ln 2 \tag{10}$$

$$= \lim_{\eta \to 0} \left(e^{\ln 2}\right)^\eta \ln 2 \tag{11}$$

$$= \ln 2. \blacksquare \tag{12}$$

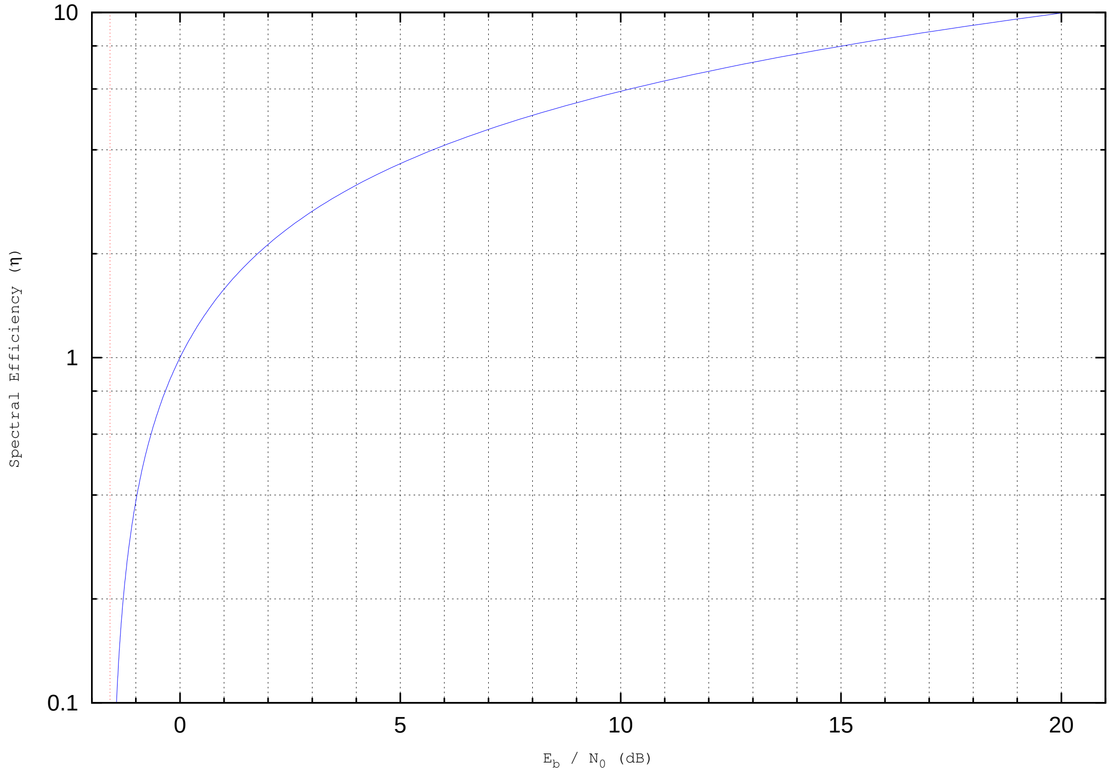

Figure 1: Capacity Curve

References

[1] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the Institute of Radio Engineers*, vol. 37, pp. 10–21, 1949.